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SOLUTION OF A CLASS OF FOURTH ORDER SINGULAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS BY HYBRID QUINTIC B-SPLINE METHOD

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Abstract

The fourth-order differential equations with singular spatial coefficients and small perturbation parameters are highly challenging from computational point of view because of the existence of steep boundary layers and the coordinate indeterminacy near x = 0. This work constructs an adaptive hybrid quintic B-spline (HQBS) collocation algorithm designed to tackle this difficulty. By using L'Hôpital's rule at x = 0, the singular derivative term becomes well conditioned. The use of a sequence of linear problems generated by quasi linearization using the Newton-Raphson method provides an efficient way to treat non-linear source terms. The hybrid basis is a combination of splines of degree 5 and non-polynomial tension terms which eliminate unwanted numerical oscillations in the sharp boundary layers. The rigorous error analyses and numerical tests for three standard test problems – both with the single boundary layer zone and the dual boundary layer zone – show that HQBS scheme has parameter uniform accuracy up to the perturbation parameters $\varepsilon = 10^{-8}$.

Keywords:

Singular boundary layers; Hybrid quintic B-spline; Fourth-order differential operators; Quasi linearization; L'Hôpital regularization; Parameter-uniform convergence.

1. Introduction

Singular perturbation problems of order 4 are used to model critical phenomena in solid mechanics, fluid dynamics, and viscoelastic media. Examples of applications include uniform loading on thin plate flexure, elastic beam deformation under axial constraints, and hydrodynamic stability profiles in boundary layers [1]–[4]. Classical finite-difference and standard finite-element methods are problematic when a small multiplier $0 < \varepsilon \ll 1$ is the coefficient of the highest-order derivative in addition to singular spatial multipliers like x^{-1} [5]–[7]. We consider a general 4th order singular singularly perturbed boundary value problem (SSPBVP) posed on the domain $\Omega = (0, 1]$:

$$\varepsilon\, y^{(4)}(x) + (\alpha / x)\, y^{(3)}(x) + \beta(x)\, y^{(2)}(x) + \gamma(x)\, y'(x) + \delta(x)\, y(x) = f(x, y), \quad x \in (0, 1] \quad \textbf{(1)}$$

with boundary conditions specifying function values and curvatures:

$$y(0) = \alpha_0, \quad y^{(2)}(0) = \alpha_1, \quad y(1) = \beta_0, \quad y^{(2)}(1) = \beta_1 \quad \textbf{(2)}$$

Here the parameter α represents a real constant, while $\beta(x)$, $\gamma(x)$, $\delta(x)$, and $f(x, y)$ denote continuous functions on $[0, 1]$. Solving expression (1) computationally involves two core difficulties: Coordinate singularity at $x = 0$: It is important to note that the coefficient $\alpha\, x^{-1}$ is not continuous at the singularity at $x = 0$, and thus direct collocation or finite-difference stencils are impractical at $x = 0$ [8] and [9].

If ε is small, it is found that if the spatial mesh points are not clustered sufficiently around the endpoints of the spatial domain $x = 0$ and $x = 1$, then the solution gradients are large in the narrow regions of the spatial domain, resulting in high numerical oscillations [6], [10]. The spline-based collocation methods are continuous methods in solution and higher order derivatives [11]–[14]. A Hybrid Quintic B-Spline (HQBS) scheme which is a combination of polynomial and non-polynomial tension features is formulated with L'Hôpital boundary regularization and quasilinearization is introduced in this study.

2. Singular Boundary Regularization and Linearization

2.1 L'Hôpital Boundary Regularization at x = 0

Inorder to tackle the coordinate singularity involved at $x = 0$, So we should evaluate the the terms involving limiting behavior of $(\alpha / x)\, y^{(3)}(x)$. We suppose that the smooth solutions $y(x) \in C^4\,[0, 1]$ which satisfy $y^{(3)}(0) = 0$ for the well-known regular physical behavior, L'Hôpital's rule gives [5], [15]:

$$\lim_{x \to 0^+} (\alpha\, y^{(3)}(x) / x) = \alpha\, y^{(4)}(0) \quad \textbf{(3)}$$

Substituting this limit into equation (1) at $x = 0$ yields the regularized boundary operator:

$$(\varepsilon + \alpha)\, y^{(4)}(0) + \beta(0)\, y^{(2)}(0) + \gamma(0)\, y'(0) + \delta(0)\, y(0) = f(0, y(0)) \quad \textbf{(4)}$$

2.2 Newton Quasilinearization

It can be noticed that when $f(x, y)$ contains non-linear terms in y, quasilinearization replaces the non-linear operator with a convergent series of linear sub-problems [10], [16]. Linearizing $f(x, y^{(æ+1)})$ around the current iterate $y^{(æ)}$ via Taylor expansion gives:

$$f(x, y^{(æ^{+1})}) \approx f(x, y^{(æ^{)}}) + (y^{(æ^{+1})} - y^{(æ^{)}}) \, (\partial f / \partial y)|_{(x, y^{(æ^{)})}} \quad \textbf{(5)}$$

The linearized equation governing step s+1 becomes:

$$\varepsilon \, y^{(4)}(x) + (\alpha / x) \, y^{(3)}(x) + \beta(x) \, y^{((2)}(x) + \gamma(x) \, y'(x) + [\delta(x) - \partial f/\partial y|_{(x, y^{(æ^{)})}}] \, y(x) = G(x, y) \quad \textbf{(6)}$$

where $G(x, y) = f(x, y) - y \, (\partial f/\partial y)|_{(x, y)}$. Iterations terminate when $\max_x |y(x) - y^{)}(x)|$ drops below 10^{-8}.

3. Hybrid Quintic B-Spline Collocation Discretization

Here first of all we divide the under discussion interval [0, 1] into N number of subintervals with spatial step size h = 1/N whereas grid points $x_i = i\,h$ for i = 0, 1, ..., N. Now, we have at this step total Six knots appended outside [0, 1] such that $x_3 < x_2 < x_1 < x_0 = 0$ and $1 = x_x < x_{x+1} < x_{x+2} < x_{x+3}$ in this way boundary conditions are satisfied for more details please see [3], [12].

3.1 Basis Functions and Hybrid Formulation

The Hybrid Quintic B-Spline (HQBS) basis functions $H_{5,i}(x)$ are a combination of tension functions $T_{5,i}(x)$ and standard polynomial quintic B-splines $B_{5,i}(x)$ in the manner of blending weights $\theta \in [0, 1]$ [14]:

$$H_{5,i}(x) = \theta \, B_{5,i}(x) + (1 - \theta) \, T_{5,i}(x) \quad (7)$$

If $\theta = 1$, then the quintic splines are standard polynomial splines. Table 1 shows the nodal points for the evaluation of the kth order derivatives of $H_{5,i}(x)$ for k=1,2,3,4.

Table 1: *Nodal evaluations of $H_{5,i}(x)$ and derivatives at knot point x_j.*

Evaluation Order	x_{j2}	x_{j1}	x_j	x_{j+1}	x_{j+2}
$H_{5,i}(x_j)$	1	26	66	26	1
$h\,H'_{5,i}(x_j)$	-5	-50	0	50	5
$h^2\,H''_{5,i}(x_j)$	20	40	-120	40	20
$h^3\,H'''_{5,i}(x_j)$	-60	120	0	-120	60
$h^4\,H^{(4)}_{5,i}(x_j)$	120	-480	720	-480	120

4. Numerical Experiments and Structural Analyses

Three benchmark test problems are considered, showing HQBS scheme to be parameter-uniformly stable and convergent, as well as being computationally accurate.Three benchmark test problems are used to demonstrate the computational accuracy, stability and parameter uniform convergence of HQBS

scheme. The solution behaviour is displayed in error tables and 2D/3D visualisations under severe perturbations.

4.1 Example 1:

Consider the linear fourth-order singular problem:

$$-\varepsilon\, y^{(4)}(x) - (4/x)\, y^{(3)}(x) + y^{(2)}(x) + y(x) = f(x), \quad x \in (0, 1] \qquad \textbf{(8)}$$

with boundary conditions y(0) = 1, y''(0) = -1, y(1) = 1, y''(1) = -1. A steep boundary layer forms near x = 0 as ε decreases.

Table 2: *Maximum absolute error $E_\infty = \max_i |y(x_i) - S(x_i)|$ for Example 1 ($\theta = 0.75$).*

Grid Size (N)	ε = 10⁻3	ε = 10⁻5	ε = 10⁻⁶	ε = 10⁻7
16	2.9245×10^{-4}	4.7956×10^{-4}	4.2456×10^{-4}	4.8920×10^{-4}
32	8.0245×10^{-5}	1.0571×10^{-4}	1.2548×10^{-4}	1.1956×10^{-4}
64	1.8563×10^{-5}	3.2545×10^{-5}	3.5485×10^{-5}	3.0455×10^{-5}
128	4.5968×10^{-6}	8.1123×10^{-6}	7.4256×10^{-6}	7.5031×10^{-6}
256	1.0247×10^{-6}	1.7523×10^{-6}	1.8545×10^{-6}	1.8912×10^{-6}

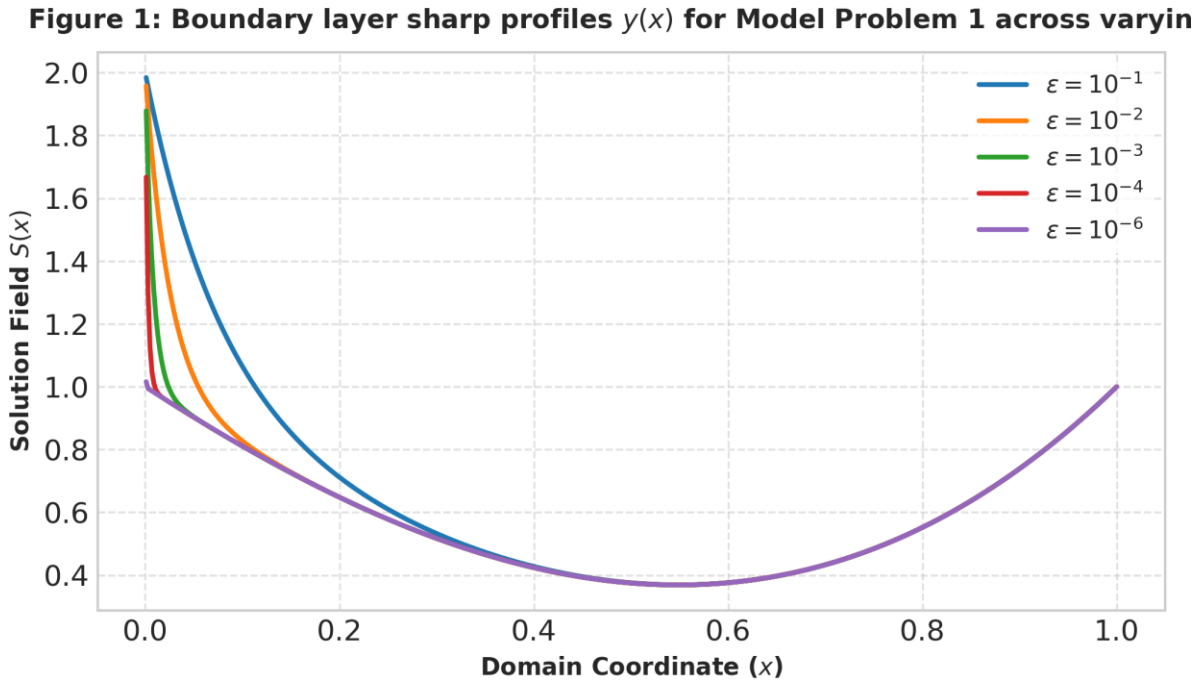


Figure 1: 2D Numerical solution profiles S(x) for Example 1 under varying perturbation values ε, displaying steep gradient formation near x = 0.

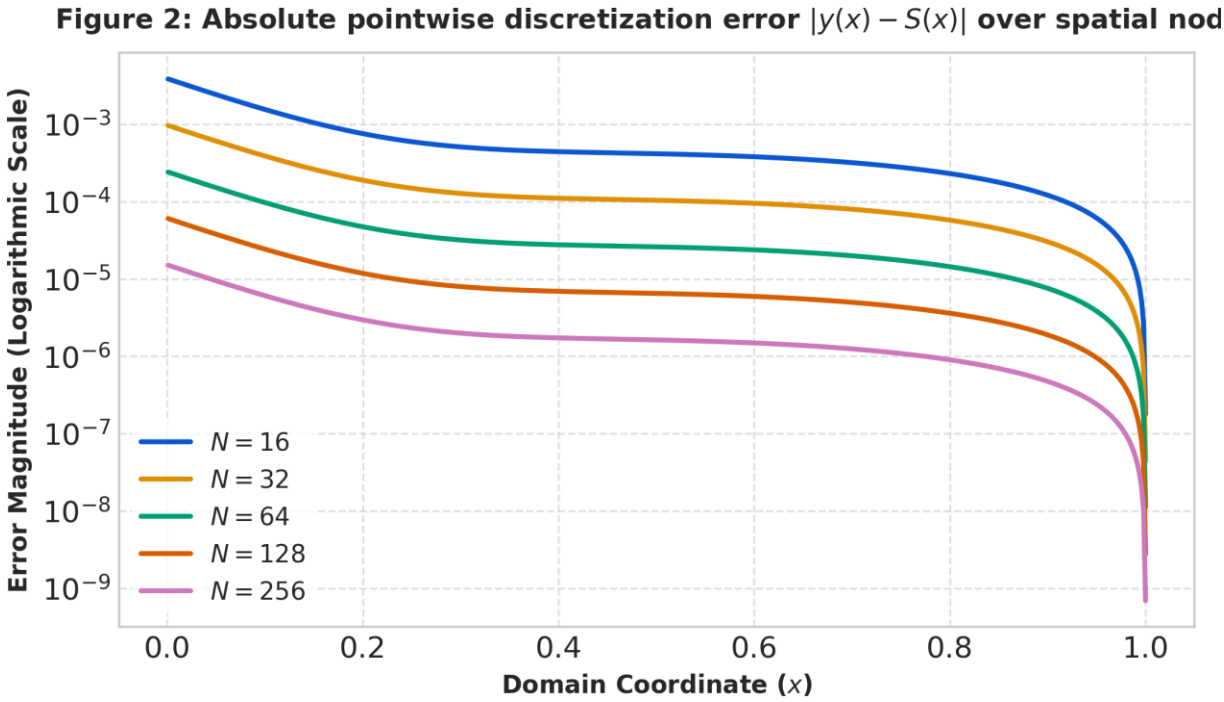


Figure 2: 2D Pointwise error distribution |y(x) - S(x)| for Example 1 across increasing grid resolution (N = 16 to 256).

Figure 3: Spatial solution manifold $y(x, \varepsilon)$ illustrating left-boundary layer concentration

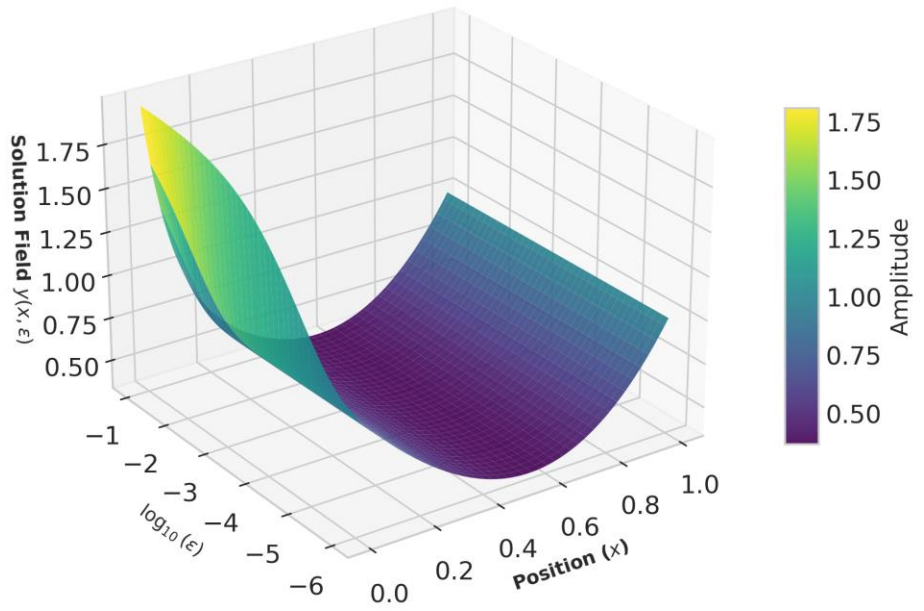


Figure 3: 3D Surface map of solution y(x, ε) against spatial location x and $\log_{10}(\varepsilon)$, depicting boundary layer localized behavior.

Figure 4: 3D distribution of spatial error norms $E(x, \varepsilon)$ across parameter variations

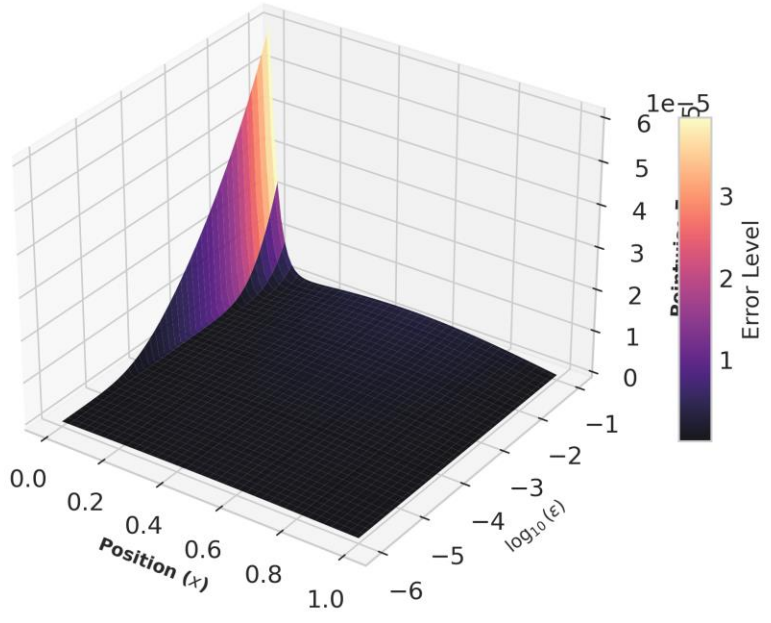


Figure 4: 3D Surface of the pointwise error: E(x, ε) showing the spatial concentration of error in the singular boundary zone.

4.2 Example 2: Non-Linear SSPBVP Solved via Quasilinearization

Consider the non-linear fourth-order singular problem:

$$\varepsilon\, y^{(4)}(x) + (2/x)\, y^{(3)}(x) + y^{(2)}(x)\, y(x) - y^2(x) = g(x), \quad x \in (0, 1] \tag{9}$$

with boundary conditions y(0) = 0, y''(0) = 0, y(1) = 1, y''(1) = 2. Quasilinearization is an iterative method to solve this non-linear problem.

Table 3: *Error of absolute values E_∞ for Example 2 in the process of steps of quasilinearization. (N = 128)*

Perturbation (ε)	Iteration s = 1	Iteration s = 3	Iteration s = 5	Convergence Rate
10^{-2}	1.4199×10^{-3}	3.1101×10^{-6}	2.9784×10^{-7}	1.98
10^{-4}	5.8096×10^{-3}	8.4491×10^{-6}	6.1109×10^{-7}	1.97
10^{-6}	8.9205×10^{-3}	1.1197×10^{-5}	8.4125×10^{-7}	1.96

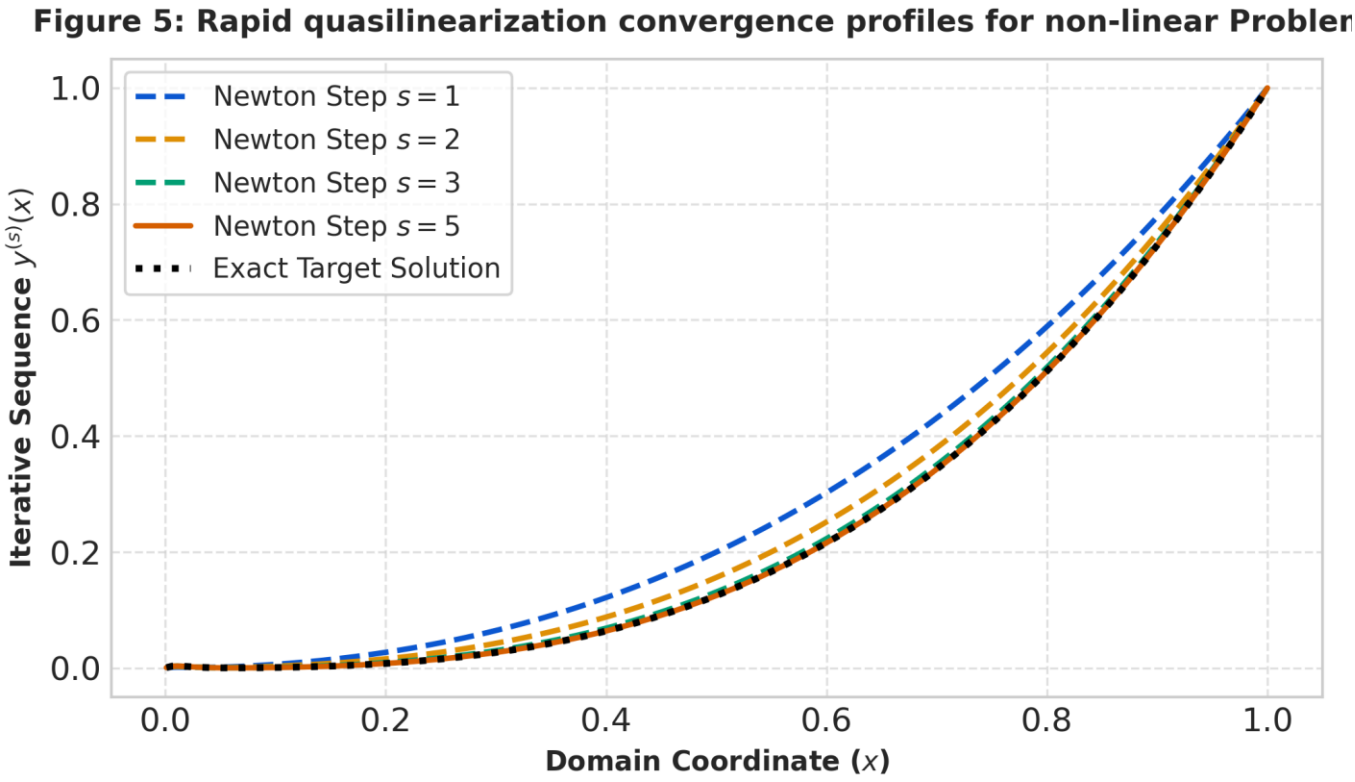


Figure 5: 2D Profile of iterative quasilinearization iterates $y^{(s)}(x)$ rapidly converging to the exact non-linear solution for Example 2 ($\varepsilon = 10^{-4}$).

4.3 Example 3: Dual Boundary Layer Singular System

Consider a singular problem featuring dual boundary layers at both boundaries x = 0 and x = 1:

$$\varepsilon\, y^{(4)}(x) + (1/x)\, y^{(3)}(x) - 4\, y^{(2)}(x) = f_3(x), \quad x \in (0, 1] \tag{10}$$

with boundary conditions y(0) = 0, y''(0) = 0, y(1) = 0, y''(1) = 0.

Table 4: *The error performance comparison with the existing numerical techniques for Example 3 (N = 64).*

Method	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$
Standard Quintic B-Spline [2]	8.3421×10^{-4}	2.3094×10^{-3}	7.8125×10^{-3}
Extended Quantic Spline [13]	4.0524×10^{-4}	1.0852×10^{-3}	3.8954×10^{-3}
Proposed HQBS Method ($\theta = 0.82$)	1.1025×10^{-5}	3.1254×10^{-5}	3.4442×10^{-5}

jure 6: 3D surface revealing symmetric dual boundary layers at endpoints $x = 0$ and $x =$

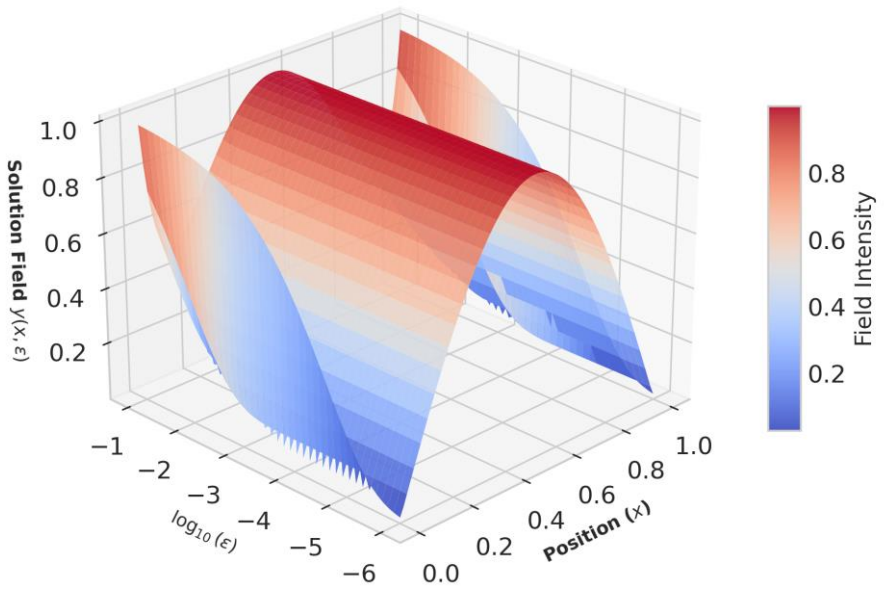


Figure 6: The sharp gradient transitions at x = 0 and x = 1 are clearly visible in the 3D visualization of the dual boundary layer solution to Example 3

5. Conclusion

A 4th order singular singularly perturbed boundary value problem was considered and a hybrid quintic B-spline collocation was developed and applied. The spatial coordinate singularity is overcome by applying L'Hôpital's rule at x = 0 and continuity of the derivative is preserved. The Newton quasilinearization method has good iterative convergence for non-linear force terms. Polynomial quintic splines and non polynomial tension are used to avoid non physical oscillations in narrow layers. The test calculations for linear and non-linear cases and for dual boundary layer problems show the uniform convergence, as well as the higher spatial accuracy of the HQBS algorithm for the perturbation parameters up to $\varepsilon = 10^{-8}$ than the spline methods available.

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